Chapter 4

Feedback Control

Control engineering is all about controlling the system's response as desired. In our RC circuit and the shock absorber, the exogenous response can be controlled by adjusting the exogenous input $v_s(t)$ and $f(t)$ respectively. We can also understand that in order to control the response we should measure the response and compare it with what is desired at that moment. The mismatch between the two values can be used to adjust the exogenous input, so that the response gradually changes towards the reference value. This mechanism is known as feedback control. In the next section we will study the basic design of feedback control systems.

4.1 Tuning Feedback Gain K

Fig.4.1 illustrates the basic feedback control mechanism in that a single gain K is used in the feedback path. The reference input, control input, system transfer function, and the response are denoted by $R(s)$, $U(s)$, $G(s)$, and $Y(s)$, respectively. Our aim is to shape up the response $Y(s)$ by adjusting K.

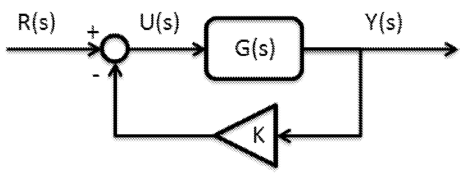


Figure 4.1: Feedback control with single feedback gain

We can derive the transfer function of the closed loop system as follows

$$
\begin{aligned}
Y(s) &= \{R(s) - KY(s)\}\, G(s) \\
\{1 + KG(s)\}\, Y(s) &= G(s)R(s) \\
G_c(s) = \frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + KG(s)}
\end{aligned} \tag{4.1}
$$

where $G_c(s)$ is the closed loop transfer function of the system. The closed loop transfer function is the open loop equivalent of the closed loop (with feedback) system as shown in Fig.4.2.

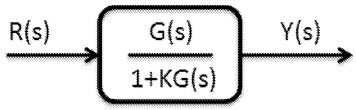


Figure 4.2: Closed loop transfer function

Lets use the closed loop transfer function (4.1) to analyze the effect of K in the response of RC circuit and shock absorber system.

4.1.1 Effect of K on Voltage Response of RC Circuit

From (3.68), the transfer function of the RC circuit is $G(s) = \frac{b}{s+a}$. From (4.1), the closed loop transfer function is

$$
\begin{aligned}
G_c(s) = \frac{Y(s)}{R(s)} &= \frac{b/(s+a)}{1 + Kb/(s+a)} \\
&= \frac{b}{s + a + Kb}
\end{aligned} \tag{4.2}
$$

The pole of the closed loop system is located at

$$
s = -(a + Kb) \tag{4.3}
$$

DC Gain

When a damped system is excited with a step input, the response attains a constant level in steady state. This phenomena is valid for both oscillatory and non-oscillatory systems. DC gain is defined as the amplitude ratio between the steady state response and step input, and can be derived as follows. Assume an exogenous input $Au_s(t)$, for which the steady state response according to the final value theorem (3.42) is

$$\begin{aligned} y(\infty) &= \lim_{s\to 0} sG(s)\frac{A}{s} \\ &= A\lim_{s\to 0} G(s) \end{aligned} \tag{4.4}$$

As $DCG = y(\infty)/A$

$$DCG = \lim_{s\to 0} G(s) \tag{4.5}$$

Then, the DC gain of the closed loop system is

$$\begin{aligned} DCG &= \lim_{s\to 0} G_c(s) \\ &= \lim_{s\to 0} \frac{b}{s+(a+Kb)} \\ &= \frac{b}{a+Kb} \end{aligned} \tag{4.6}$$

From (4.3) and (4.6) we can understand that as K increase, the pole moves towards $-\infty$, and $DCG \to 0$ (response gets weaker). Lets analyze the effect of K in more details for that we assume $R_1 = R_2$ for the sake of simplicity. Then, $a = \frac{R_1+R_2}{R_1 R_2 C} = \frac{2}{RC}$, and $b = \frac{1}{RC}$. We can identify few specific conditions as follows.

1. $K = 0$ brings the closed loop system to original open loop system where $s = -a$

2. The value of K for closed loop pole to be located at the origin $s = 0$ is $K = -\frac{a}{b} = -2$, in which case the response (4.2) is

$$\begin{aligned} Y(s) &= \frac{b}{s+a-2b} R(s) \\ &= \frac{1/RC}{s+2/RC-2/RC} R(s) \\ &= \frac{1}{RC}\frac{1}{s} R(s) \end{aligned} \tag{4.7}$$

This response in time domain as shown in (4.8) is known as rigid body dynamics.

$$y(t) = \frac{1}{RC}\int r(t)dt \tag{4.8}$$

3. When the DC gain of the system is unity

$$\begin{aligned} DCG = \frac{b}{a+Kb} &= 1 \\ K &= 1-\frac{a}{b} \\ &= -1 \end{aligned} \tag{4.9}$$

4. $K < -2$ locates the pole on the positive half plane $s > 0$, producing an exponential term e^{at} in the response. This term grows indefinitely causing the system unstable.

We are also curious to know how the control input $u(t)$ varies with time. For that, we reconfigure the system block diagram in Fig.4.2 as shown in Fig. 4.3 where $U(s)$ appears as the output. Block diagram algebra (Appendix D) allows us to manipulate blocks in this way so that to find transfer functions between any two signals of the system.

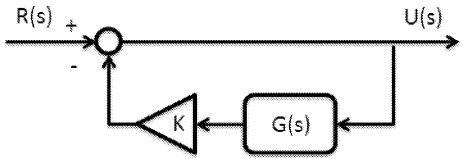


Figure 4.3: Monitoring control input of the feedback system

$$\begin{aligned} \frac{U(s)}{R(s)} &= \frac{1}{1+Kb/(s+a)} \\ &= \frac{s+a}{s+a+Kb} \end{aligned} \tag{4.10}$$

In Matlab we can simulate both response (4.2) and control input (4.10) for various values of K as shown in Fig.4.4. In these simulations $R = 1\text{k}\Omega$ and $C = 200\mu\text{F}$ were assumed. These results show that feedback gain K can be manipulated in order to shape up the response as we wish. Let us see how K can be used to achieve a desired time constant (Appendix E) in the response. From (4.2), unit step response is given by

$$\begin{aligned} Y(s) &= \frac{b}{s+(a+Kb)}\frac{1}{s} \\ &= \frac{b}{a+Kb}\left[\frac{1}{s} - \frac{1}{s+(a+Kb)}\right] \end{aligned}$$

After inverse Laplace transformation

$$y(t) = \frac{b}{a + Kb}\left[1 - e^{-(a+Kb)t}\right] \tag{4.11}$$

The time constant of the system is therefore

$$\tau = \frac{1}{a + Kb} \tag{4.12}$$

Then, the value of K can be calculated as follows in order to achieve a desired time constant.

$$K = \frac{1}{b}\left[\frac{1}{\tau} - a\right] \tag{4.13}$$

For the values assumed earlier as $R = 1\text{k}\Omega$ and $C = 200\mu\text{F}$, two parameters a and b assume values as $a = 2/(RC) = 10$, and $b = 1/(RC) = 5$. Lets assume that we need to achieve a time constant $\tau = 50\text{ms}$. The feedback gain required for that is $K|_{\tau=0.050} = [1/50 \times 10^{-3} - 10]/5 = [20 - 10]/5 = 2$. From (4.9)the DC gain at this condition $DCG|_{K=2} = \frac{b}{a+Kb} = 0.25$. Once the time constant is achieved, we can adjust the steady state level by using a forward gain of $1/DCG$ as shown in Fig.4.5. Once time constant and DC gain has been adjusted, the response is shown in Fig.4.6

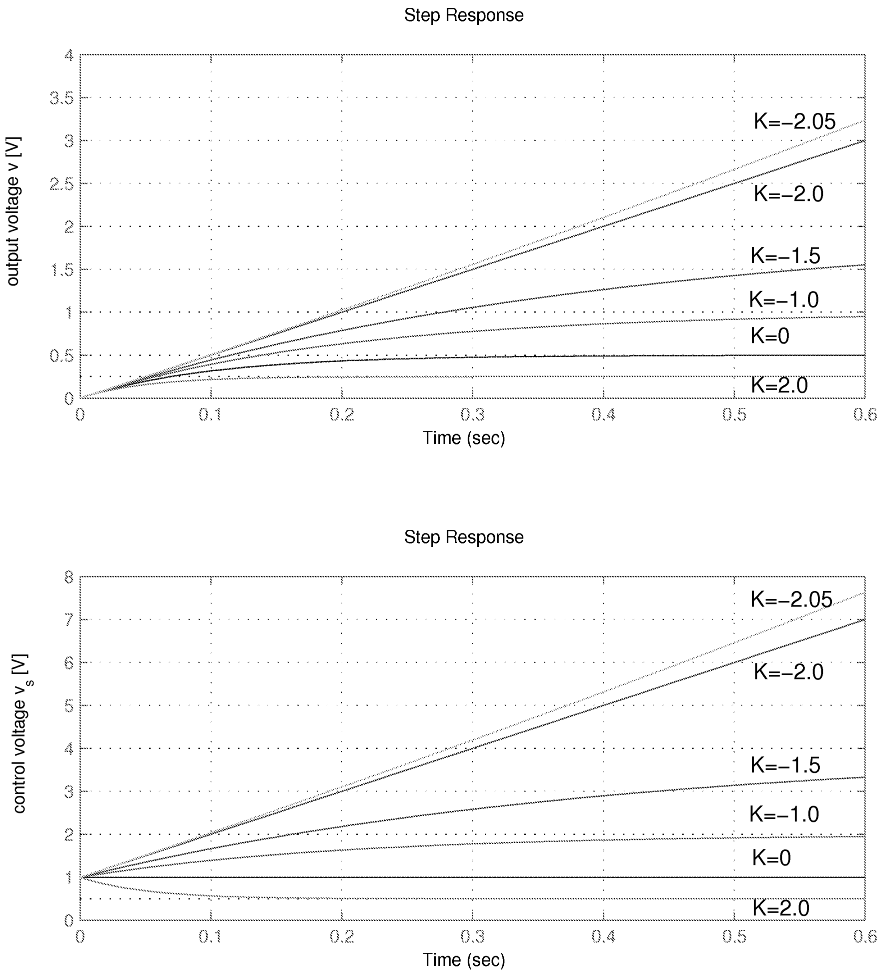


Figure 4.4: System response and control input of the RC circuit

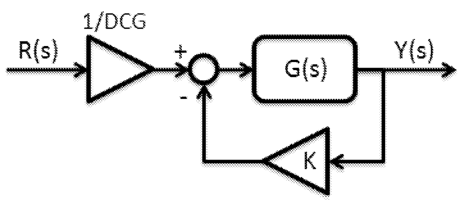


Figure 4.5: DC gain adjustment by use of a forward gain

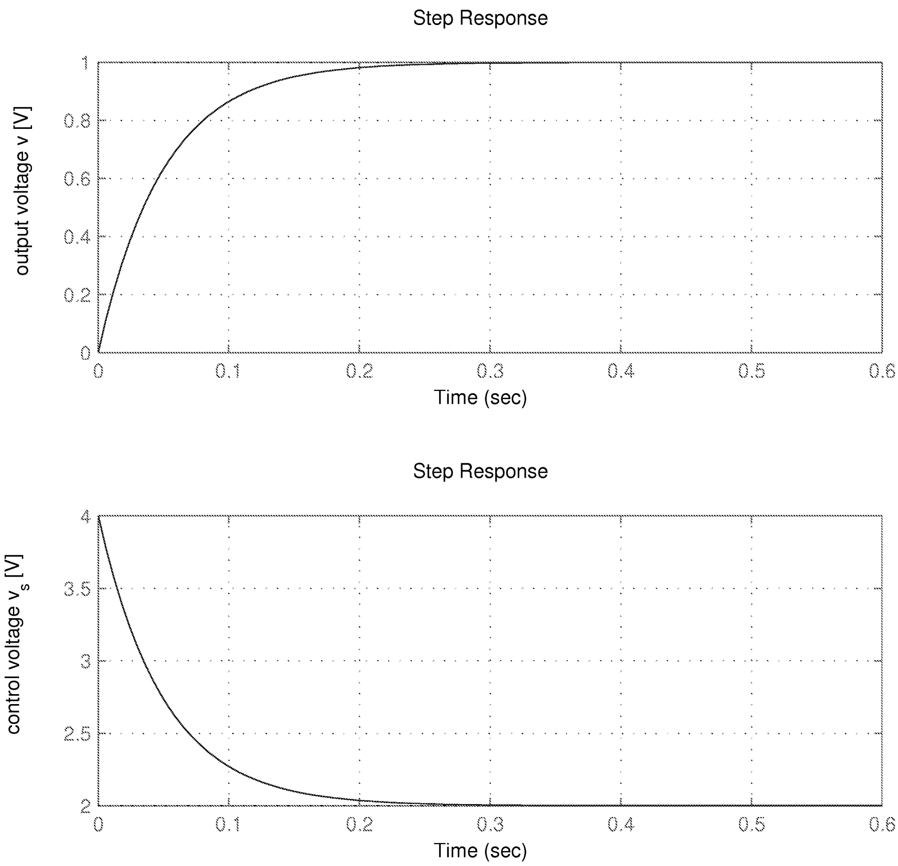


Figure 4.6: Simulation result after time constant and DC gain have been adjusted

4.1.2 Effect of K on Height Control of Spring-Damper System

Our Shock absorber can be used to design a height control table in that we control the exogenous force $f(t)$ in such a way that system response $y(t)$ eventually reaches a desired reference height $y_r(t)$. In this design also, lets use a simple feedback gain K and construct a test control system as shown in Fig.4.7

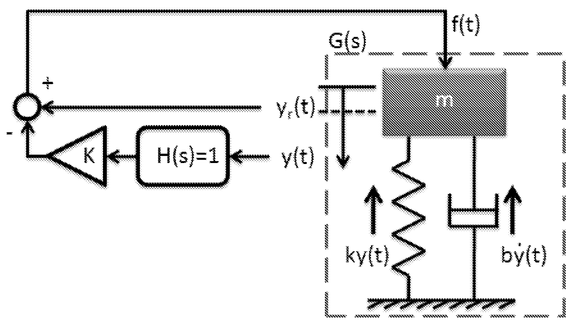


Figure 4.7: Height control of a shock-absorber table using a feedback gain

From (3.69), the transfer function of the shock absorber is $G(s) = \frac{\eta}{s^2+2\sigma s+\rho}$ where $\eta = 1/m, \sigma = \frac{b}{2m}$ and $\rho = k/m$. According to (4.1), the closed loop transfer function is

$$G_c(s) = \frac{\eta/(s^2+2\sigma s+\rho)}{1+K\eta/(s^2+2\sigma s+\rho)} \quad (4.14)$$

$$= \frac{\eta}{s^2+2\sigma s+\rho+K\eta} \quad (4.15)$$

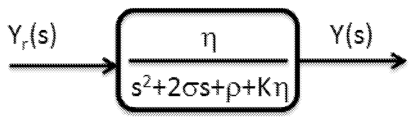


Figure 4.8: Height control system with a single feedback gain K

The characteristic equation $\Delta(s) = s^2 + 2\sigma s + \rho + K\eta = 0$ is affected by the feedback gain and the closed loop poles of the system are deflected as follows

$$s_1, s_2 = -\sigma \pm \sqrt{\sigma^2 - \rho - K\eta} \quad (4.16)$$

This leads to a few interesting scenarios as follows.

When K is lowered sufficiently that $\sqrt{\sigma^2 - \rho - K\eta} = \sigma$, one of the two poles is located at the origin, thus the system is at the verge of instability. We assign $K = K_u$ for this condition where

$$K_u = -\frac{\rho}{\eta} \tag{4.17}$$

When $K = 0$ system returns to the original open loop system. And, when $\sigma^2 - \rho - K_c\eta = 0$ two poles coincide each other. We assign $K = K_c$ for this condition where

$$K_c = \frac{\sigma^2 - \rho}{\eta} \tag{4.18}$$

Referring to (4.17) and (4.18) we can describe how K affects the behavior of the control system as follows

1. when $K < K_u$ closed loop system is unstable.
2. when $K_u < K < K_c$ the closed loop system has two distinct real poles (over damped).
3. when $K = K_c$ closed loop system is critically damped (fastest response).
4. when $K > K_c$ closed loop system has a pair of complex conjugate poles (oscillatory stable).

We also notice that the closed loop system DC gain is

$$DCG = \frac{\eta}{\rho + K\eta} \tag{4.19}$$

Therefore, we can see that as K increases $DCG \to 0$. Figure 4.9 illustrates how the response is affected as K changes.

We are also interested in knowing how the force changes under the feedback control. For that, we can alter the block diagram in Fig.4.7 as shown in Fig.4.10 so that the force appears as the output.

Referring to (4.1) and Fig.4.10, the force input is

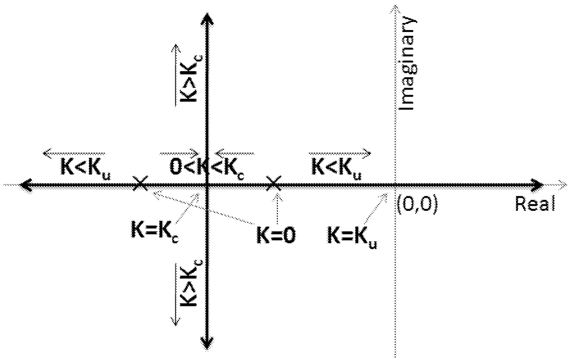


Figure 4.9: Trajectory of the closed loop poles as K changes

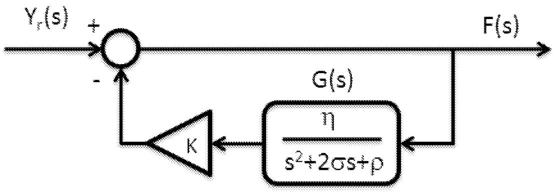


Figure 4.10: Altered block diagram to monitor the force

$$\frac{F(s)}{Y_r(s)} = \frac{1}{1 + K\eta/(s^2 + 2\sigma + \rho)} \tag{4.20}$$

$$= \frac{s^2 + 2\sigma + \rho}{s^2 + 2\sigma + \rho + K\eta} \tag{4.21}$$

4.1.3 Simulation

Lets assume m=50[kg], b=700[Ns/cm], and k=125[N/cm]. Then, σ=7, ρ=2.5, and η=0.02. We calculate K_c=2325, and K_u=-125. We simulate the unit step (1cm) response for $K = K_c + 6000$, $K = K_c$, and $K = (K_c + K_u)/2$ and the results are shown in Fig.4.11. Its generally desirable to obtain critically damped response, as it gives the fastest settling time. When the closed loop system is critically damped DC gain is given by

$$DCG_c = \frac{\eta}{\rho + K_c\eta}$$

$$= \frac{\eta}{\rho + \frac{\sigma^2 - \rho}{\eta}\eta}$$

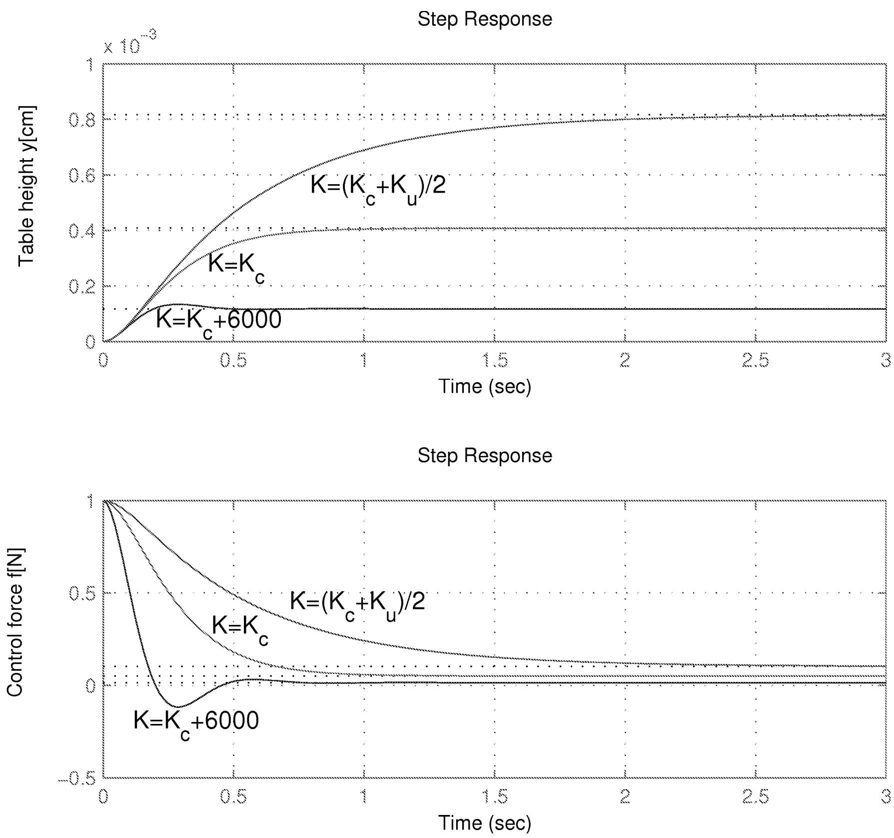


Figure 4.11: Table height and force input under various feedback gains

$$= \frac{\eta}{\sigma^2} \tag{4.22}$$

In this example $DCG_c = 4.082 \times 10^{-4}$[cm] is far too low. In order to lift the response to the desired level of 1[cm] we could amplify the reference input by a gain $1/4.082 \times 10^{-4} = 2450$ as shown in Fig.4.12. With this modification, we get the system to respond as shown in Fig.4.13.

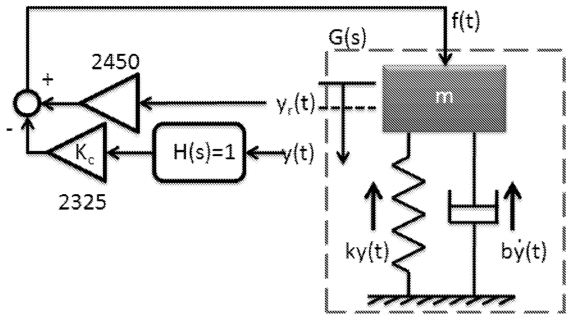


Figure 4.12: Modifying reference input to compensate for DC gain

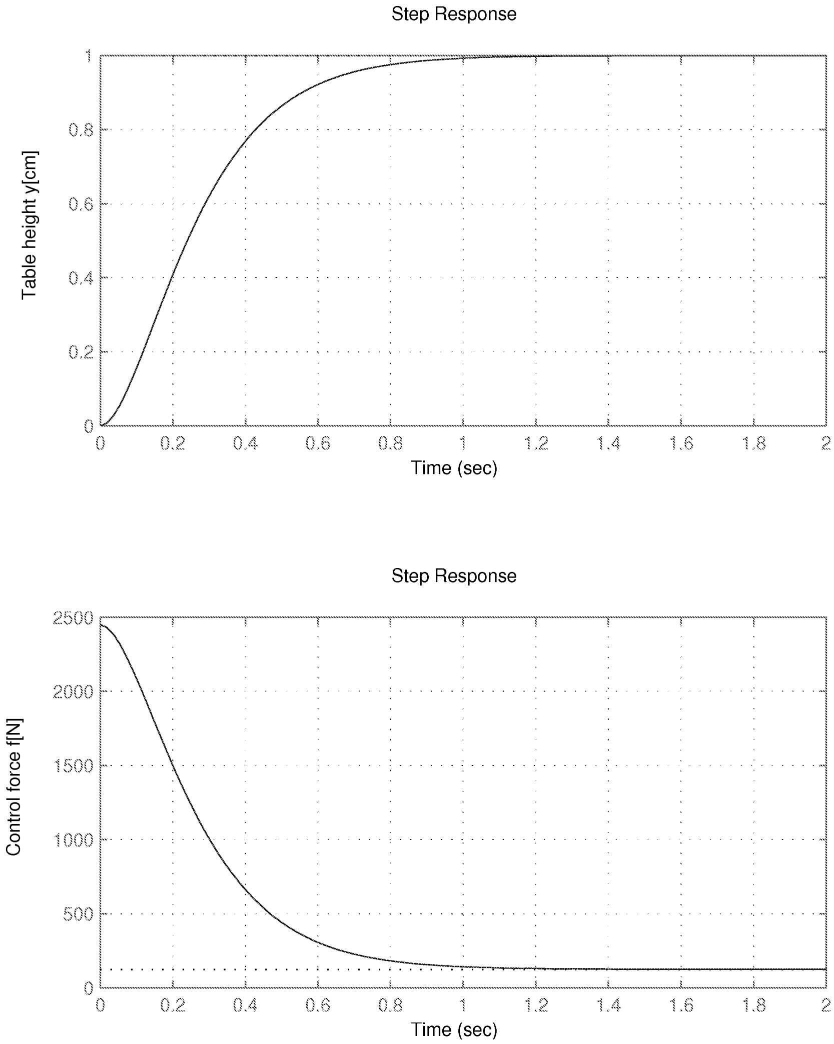


Figure 4.13: Response with critically tuned feedback gain and DC gain compensated reference input

4.2 Root Locus Design

We have already seen that a feedback gain can be used to move closed loop poles of a feedback control system, thus to shape up the response. This method of using a gain to move closed loop poles was first proposed by Evans [5]. In this section, lets formally analyze the shape of the path along which closed loop poles move as feedback gain increases. Lets consider a feedback control system with a feedback gain K and a sensor transfer function $H(s)$ as shown in Fig.4.14.

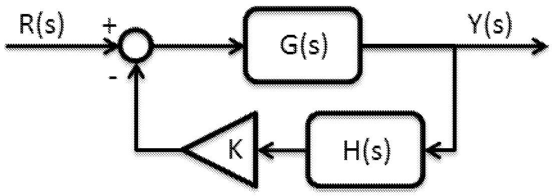


Figure 4.14: Feedback control with feedback $H(s)$ and feedback gain K

we can write following equations and derive the closed loop transfer function as follows.

$$\begin{aligned} Y(s) &= G(s)\{R(s) - KH(s)Y(s)\} \\ Y(s)\{1 + KH(s)G(s)\} &= G(s)R(s) \\ \frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + KH(s)G(s)} \end{aligned} \tag{4.23}$$

Then, the characteristic equation $\Delta(s)$ is

$$\begin{aligned} 1 + KH(s)G(s) &= 0 \\ KH(s)G(s) &= -1 \end{aligned} \tag{4.24}$$

which shows that the system poles can be moved by changing K. The path of closed loop poles is known as root locus. From (4.24) root locus must satisfy following conditions

$$|KG(s)H(s)| = 1 \tag{4.25}$$

$$\angle KH(s)G(s) = \pm 180^o(2k+1) \tag{4.26}$$

where $k = 0, 1, 2, ...$

4.2.1 Root Locus Rules

Rule 1: Root locus is on the real axis to the left of an odd number of poles or zeros
Using pole-zero form of the transfer function, the angle condition (4.26), can be written as follows

$$\begin{aligned}\angle K\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} &= \pm 180^o(2k+1)\\ \angle(s-z_1)+\angle(s-z_2)+\dots+\angle(s-z_m)&\\ -\angle(s-p_1)-\angle(s-p_2)-\dots-\angle(s-p_n) &= \pm 180^o(2k+1) \quad (4.27)\end{aligned}$$

where z_i and p_i are ith zero and ith pole of $H(s)G(s)$.

Angle of a zero, a pole, and a complex conjugate pair of zeros/poles on the real axis: A real zero z with respect to any point on the real axis s creates an angle as follows.

$$\angle(s-z) = \begin{cases} 0 & \text{if} \quad s \geq z \\ 180^o & \text{if} \quad s < z \end{cases} \quad (4.28)$$

which says that a real zero does not have any angle contribution on the real axis to the right of it. And, it has 180^o angle contribution on the real axis to the left of it. Similarly, a real pole p with respect to any point on the real axis s creates an angle as follows.

$$-\angle(s-p) = \begin{cases} 0 & \text{if} \quad s \geq p \\ -180^o & \text{if} \quad s < p \end{cases} \quad (4.29)$$

which says that a real pole does not have any angle contribution on the real axis to the right of it. And, it has -180^o angle contribution on the part[of the real axis to the left of it. Lets work out the same for a complex conjugate pair of poles $p_1, p_2 = a \pm jb$ or zeros $z_1, z_2 = a \pm jb$. The angle contribution at a point s on the real axis by a pair of complex conjugate poles is illustrated in Fig.4.15 in that the angle contribution is as follows.

$$-\angle(s-a+jb)-\angle(s-a-jb) = \begin{cases} -(360^o-\theta)-\theta = -360^o \\ \text{if} s \geq a \\ -(180^o+\theta)-(180^o-\theta) = -360^o \\ \text{if} s < a \end{cases} \quad (4.30)$$

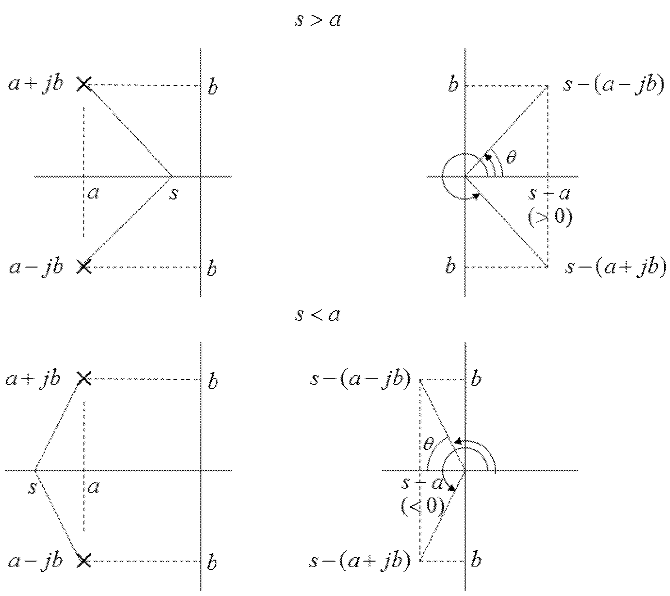


Figure 4.15: Angle contribution on the real axis by a complex conjugate pole pair

where $\theta = \tan^{-1}\frac{b}{|s-a|}$. Therefore, a complex conjugate pair of poles does not make any angle contribution on the real axis. Same relationship can be shown for a complex conjugate pair of zeros where the angle contribution is always zero. From (4.28), (4.29), and (4.30), we can conclude that on the real axis to the right of a pole or a zero there is no angle contribution. And, a complex conjugate pair of poles or zeros also do not have any angle contribution. There is a non zero angle contribution by a pole or a zero only on the real axis segment to the left of them. For real zero this contribution is 180^o, whereas for a real pole its -180^o. Therefore, the angle condition (4.26) is verifiable.

Rule 2: Root locus starts from open loop poles (set $K \to 0$), and arrive at open loop zeros (set $K \to \infty$)

We can write root locus condition (4.24) using zeros and poles of the open loop transfer function as follows

$$\begin{aligned} 1 + K\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} &= 0 \\ (s-p_1)(s-p_2)\ldots(s-p_m) + K(s-z_1)(s-z_2)\ldots(s-z_m) &= 0 \end{aligned}$$

$$ \tag{4.31} $$

When $K = 0$, characteristic equation becomes as follows.

$$\Delta(s) = (s - p_1)(s - p_2)\dots(s - p_m) \rightarrow 0 \tag{4.32}$$

which shows that the close loop poles are as same as the open loop poles $p_1, p_2, \dots p_n$. When $K \rightarrow \infty$, the requirement to satisfy (4.31) is

$$(s - z_1)(s - z_2)\dots(s - z_m) \rightarrow 0 \tag{4.33}$$

showing that closed loop poles getting closer to the open loop zeros $z_1, z_2, \dots z_m$.

Rule 3: Root locus asymptotes, asymptote angles, and point of intersection

Out of n poles, m of them terminate at open loop zeros. The other $n-m$ poles asymptotically reach infinity. We have seen this in the height control table, where Fig. 4.9 shows the two poles approaching infinity vertically upwards and downwards. Therefore, root locus should have $n-m$ asymptotes. These asymptotes intersect each other at a common point on the real axis $x = \alpha$ and they make ϕ_i angles with the real axis as given by

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{n - m} \tag{4.34}$$

$$\phi_l = \frac{180^o + 360(l-1)}{n - m} \quad l = 0, 1, \dots (n - m) \tag{4.35}$$

Rule 4: Angle of departure/arrival

Starting from the open loop poles, root locus starts along a certain direction. For each open loop pole, this angle is known as the angle of departure. Similarly, for each zero, root locus arrives at an angle, which is known as the angle of arrival. Figure 4.16 illustrates a pole p_i from which root locus starts with angle of departure θ_i. A point s on the root locus makes an angle $\angle(s - p_i)$, and as $s \rightarrow p_i$, $\angle(s - p_i) \rightarrow \theta_i$.
A very close point s^* to the open loop pole p_i on the root locus must satisfy the angle condition as follows.

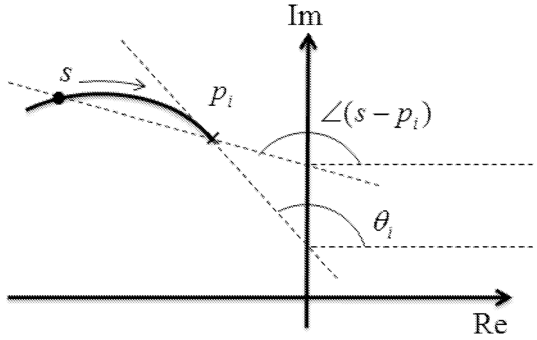


Figure 4.16: Root locus departure angle from a pole

$$\begin{aligned}
\angle G(s)|_s &= \pm 180^o(2k+1) \\
\angle(s^* - z_1) + \angle(s^* - z_2) + \ldots + \angle(s^* - z_m) & \\
-\angle(s^* - p_1) - \angle(s^* - p_2) - \ldots - \angle(s^* - p_i) & \\
- \ldots - \angle(s^* - p_n) &= \pm 180^o(2k+1) \quad (4.36)
\end{aligned}$$

where k=0,1,2...As $s^* \approx p_i$. we can write (4.36) as follows

$$\begin{aligned}
\angle(p_i - z_1) + \angle(p_i - z_2) + \ldots + \angle(p_i - z_m) & \\
-\angle(p_i - p_1) - \angle(p_i - p_2) - \ldots - \theta_i & \\
- \ldots - \angle(p_i - p_n) &\approx \pm 180^o(2k+1) \quad (4.37)
\end{aligned}$$

where θ_i is the angle of departure of the pole p_i. This tells us that in order to calculate the angle of departure of a pole, or the angle of arrival of a zero can be calculated by substituting that pole or zero onto (4.26).

Rule 5: Break away/in point
Closed loop poles may move towards each other, and meet, and then depart from each other. The location where poles meet each other must satisfy the condition given below without the proof.

$$\begin{aligned}
\frac{d}{ds}\left\{\frac{1}{H(s)G(s)}\right\} &= 0 \\
\frac{d}{ds}\left\{\frac{D_{HG}(s)}{N_{HG}(s)}\right\} &= 0 \quad (4.38)
\end{aligned}$$

where $H(s)G(s) = \frac{N_{HG}(s)}{D_{HG}(s)}$.

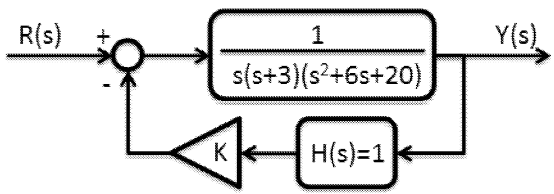
Rule 6: Stability margin
The characteristic equation in (4.31) can be put onto the Routh array and determine the range of gain K for stable response (Note: We'll work out an example to demonstrate Routh stability criterion)

Rule 7: Gain calculation
At any pole location, the corresponding gain can be determined by the root locus gain condition (4.25). In order to locate poles at $s = s^*$ the required gain gain $K = K^*$ can be determined as follows

$$\begin{aligned} K^* &= \frac{1}{|H(s)G(s)|_{s=s^*}} \\ K^* &= \left|\frac{D_{HG}(s)}{N_{HG}(s)}\right|_{s=s^*} \end{aligned} \tag{4.39}$$

4.2.2 Example

A feedback control system is shown below.



Poles: The characteristic equation is $1 + K\frac{1}{s(s+3)(s^2+6s+20)} = 0$. When $K = 0$, characteristic equation of the system is $s(s+3)(s^2+6s+20) = 0$. Thus, the system has four poles ($n = 4$) located at $0, -3, -3 \pm j\sqrt{11}$. According to rule 1, root locus is on the real axis in the range $[-3, 0]$. There are no open loop zeros ($m = 0$), therefore, there should be $n - m$=4 asymptotes as $K \to \infty$.

Asymptotes: The four asymptotes makes angles with the real axis

$$\phi_0 = \frac{180^0 + 360^0(0-1)}{4} = -45^0$$

$$\begin{aligned}\phi_1 = \frac{180^0 + 360^0(1-1)}{4} &= 45^0 \\ \phi_2 = \frac{180^0 + 360^0(2-1)}{4} &= 135^0 \\ \phi_3 = \frac{180^0 + 360^0(3-1)}{4} &= 225^0\end{aligned}$$

And they intersect each other on the real axis at

$$\alpha = \frac{(0) + (-3) + (-3 + j\sqrt{11}) + (-3 - j\sqrt{11})}{4} = -2.25$$

Angle of departure: Lets consider the open loop pole at $-3 + j\sqrt{11}$. If departure angle θ is given by

$$\begin{aligned} -\angle(-3 + j\sqrt{11} - 0) - \angle(-3 + j\sqrt{11} + 3) & \\ -\theta - \angle(-3 + j\sqrt{11} - (-3 - j\sqrt{11}) &= 180^0(2k+1) \\ -\left(180^0 - \tan^{-1}\frac{\sqrt{11}}{3}\right) - 90^0 - \theta - 90^0 &= 180^0(2k+1) \\ 48^0 - \theta &= 180^0 \\ \theta &= -132^0 \end{aligned}$$

Due to symmetry, the other conjugate pole has a departure angle of 132^0.

Breakaway point: The two real poles meet each other on the real axis at a point, which satisfies condition (4.38). As

$$\begin{aligned} \frac{d}{ds}\left\{\frac{1}{H(s)G(s)}\right\} &= 0 \\ \frac{d}{ds}\{s(s+3)(s^2+6s+20)\} &= 0 \\ \frac{d}{ds}\{s^4 + 9s^3 + 38s^2 + 60s\} &= 0 \\ 4s^3 + 27s^2 + 76s + 60 &= 0 \end{aligned}$$

This expression can be numerically solved using Newton-Raphson method (Appendix C) as follows. Lets say $f(s) = 4s^3 + 27s^2 + 76s + 60$. We also know that the two real poles at 0 and -3 will meet somewhere in [-3,0]. And, we also can know that $f(-2) < 0$ and $f(-1) > 0$, therefore, the breakaway point must be in the range [-1,-2]. Lets guess initial solution as s_0 =-1.5, and use Newton-Raphson iteration as follows

$$\begin{aligned}
s_1 &= s_0 - \frac{f(s_0)}{f'(s_0)} \\
&= s_0 - \frac{4s^3+27s^2+76s+60}{12s^2+54s+76}|_{s_0} \\
&= -1.5 - \frac{4\times -1.5^3 + 27\times -1.5^2 + 76\times -1.5 + 60}{12\times -1.5^2 + 54\times -1.5 + 76} \\
&= -1.5 - \frac{-6.75}{22} \\
&= -1.19
\end{aligned}$$

Lets do a second iteration as follows

$$\begin{aligned}
s_2 &= s_1 - \frac{f(s_1)}{f'(s_1)} \\
&= -1.19 - \frac{4\times -1.19^3 + 27\times -1.19^2 + 76\times -1.19 + 60}{12\times -1.19^2 + 54\times -1.19 + 76} \\
&= -1.19 - \frac{1.05}{28.73} \\
&= -1.19 - 0.04 \\
&= -1.23
\end{aligned}$$

Lets stop iteration here and say that the breakaway point is at $s = -1.23$.

According to rule 7 in (4.39) and knowing that $N_{HG}(s) = 1$ the gain at the breakaway point is $K = |D_{HG}(s)|_{s=-1.23} = |-1.23(-1.23+3)(-1.23^2+6\times -1.23+20)| = |-30.77| \approx 30.8$.

Stability margin: From (4.31) the characteristic equation is $|\Delta(s) = s(s+3)(s^2+6s+20)+K = 0$. By expanding it $\Delta(s) = s^4+9s^3+38s^2+60s+K = 0$, which can be inserted to the Routh array as shown in Fig.4.2.2.
It can be proven that when the first coefficient on the array becomes negative, the system poles cross the imaginary axis and move into the positive half of the s-plane. At this point system looses stability, thus the condition for stability is $60 - 0.29K \geq 0$, which tells that $K \leq 208$.

MatLab simulation: Following code in MatLab will draw the root locus of the above example as shown in Fig.4.18. We can also find the close loop

s^4	1	38	K
s^3	9	60	0
s^2	(9×38-60×1)/9 = **31.3**	(60K-0×38)/60 = K	0
s^1	(31.3×60-9×K)/31.3= **60-0.29K**	**0**	
s^0	[(60-0.29K) ×K-0×31.3]/60-0.29K = **K**		

Figure 4.17: Routh array of the characteristic equation

$0 \leq K < 30.8$	Two distinct real poles and a pair of stable complex conjugate poles
K=30.8	Two real coincident poles and a pair of stable complex conjugate poles
$30.8 < K \leq 209$	Two pairs of stable complex conjugate poles
$209 < K$	Stable and unstable pairs of complex conjugate poles

Table 4.1: Nature of poles as feedback gain K increases

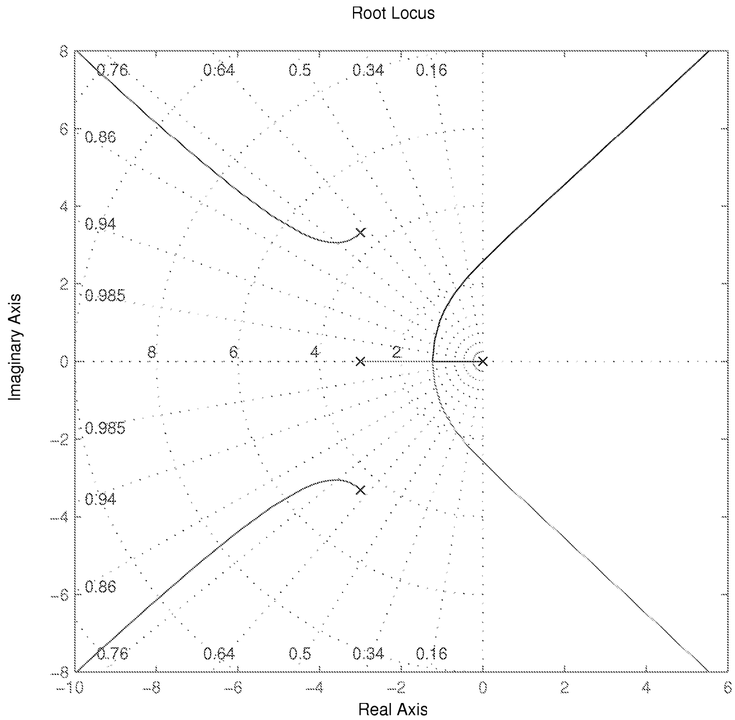
system performance and corresponding gain by clicking on any point on the root locus.

```
% Plant OLTF
num=[1]; den=conv([1 0],conv([1 3],[1 6 20]));
G=tf(num,den);
rlocus(G);
grid on;
```

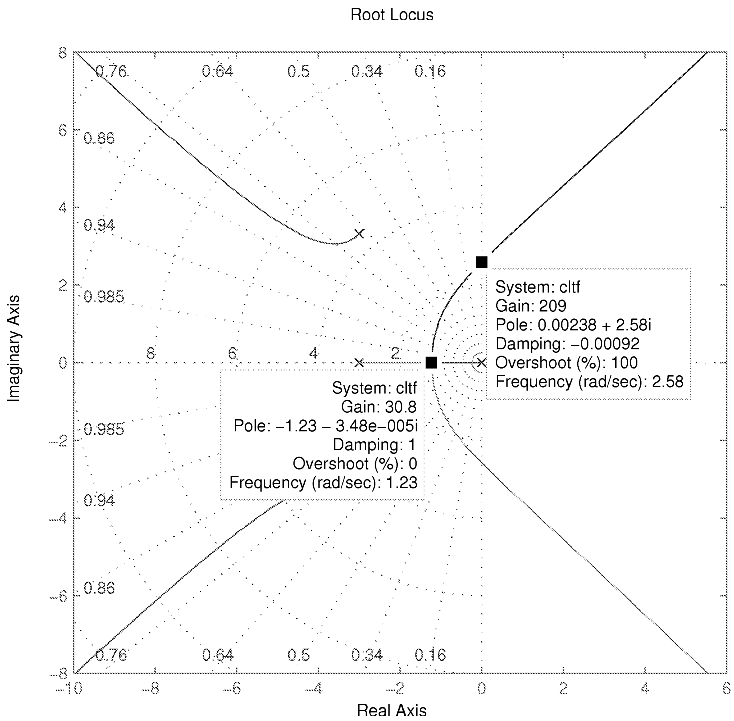
From the root locus, we can know that the two real poles become a pair of complex conjugate poles when $K > 30.8$, and the same two poles become unstable when $K > 209$. The nature of the four poles as K increases is listed in the table 4.1.

We can use the following MatLab codes to determine the unit step response of the closed loop plant $\frac{G(s)}{1+KH(s)G(s)}$ for some important values of K=20(over damped), 30(critically damped), 150(under damped), and 240 (unstable).

```
% Plant OLTF
num=[1]; den=conv([1 0],conv([1 3],[1 6 20]));
```



(a)



(b)

Figure 4.18: (a) Root locus of $1 + KH(s)G(s) = 0$, (b) Gains at breakaway point and when the system is on the verge of instability

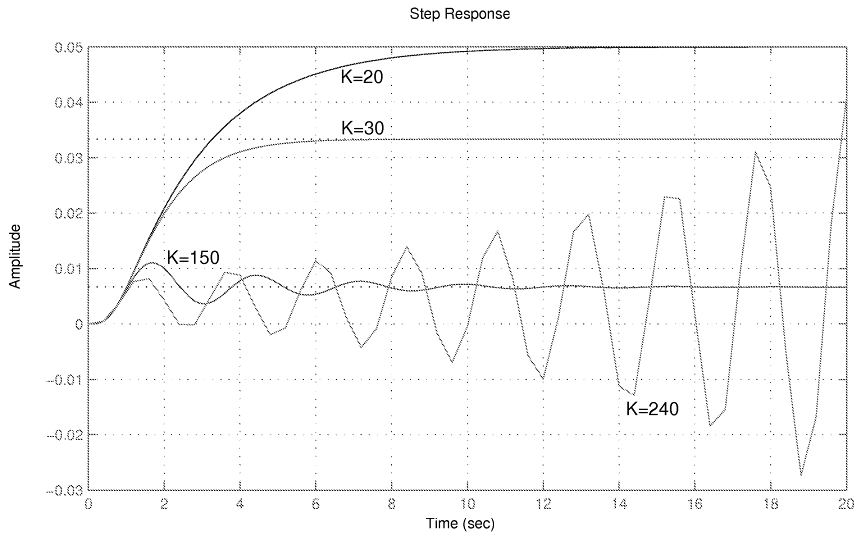


Figure 4.19: Various responses of the closed loop system for a unit step input

```
G=tf(num,den);
dur=20; % simulation duration 20s

% Responses 1:  over damped
K=20;
cltf1=feedback(G,K);
step(cltf1,dur); hold on;

% Responses 2:  critically damped
K=30;
cltf2=feedback(G,K);
step(cltf2,dur); hold on;

% Responses 3:  under damped
K=150;
cltf3=feedback(G,K);
step(cltf3,dur); hold on;

% Responses 4:  unstable
K=240; cltf4=feedback(G,K);
step(cltf4,dur); hold on;
```

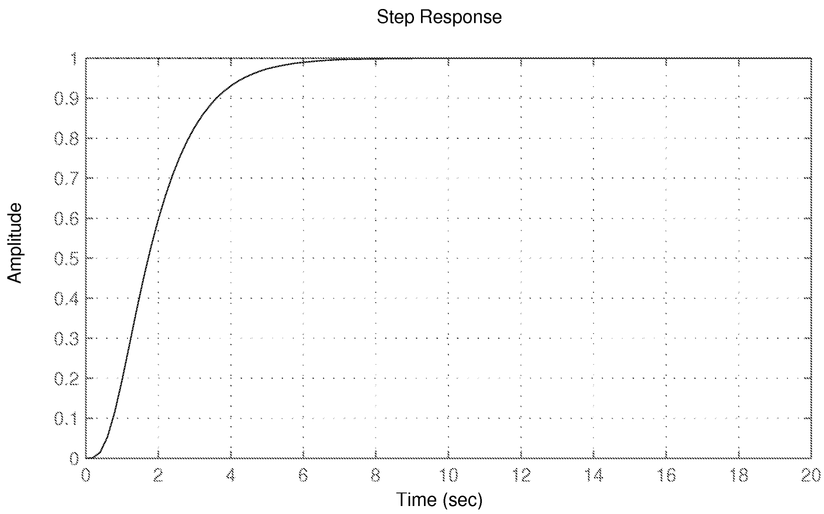


Figure 4.20: Response with $K = 30$ and a pre-gain adjusted for $DCG = 1$

It is quite evident that a range of controlled responses can be obtained by manipulating gain K, thus, the control engineer can set the most appropriate value of K in order to achieve the desired response. Various values for K can be used for which system response can be simulated, and then, select the best value for K. For instance, lets select $K = 30$, then the DCG of the closed loop system is $DCG = \lim_{s\to 0} \frac{N_{HG}(s)}{D_{HG}(s)+K}|_{s=0} = \frac{1}{s^4+9s^3+38s+60s+K}|_{s=0} = \frac{1}{K}$, which means that the steady state value of the response is $\frac{1}{K}$ times the step input magnitude. Therefore, we can now use a pre-gain of $\frac{1}{DCG} = K$ as shown in Fig.4.20 to improve the steady state response to match the reference input magnitude.

A good set of examples on root locus design can be found in [6] in that the gain K is placed in the forward path.

4.3 Summary

The response of a system can be controlled by sensing it through a sensor $H(s)$, amplifying it using a gain K and comparing with the desired value $R(s)$, and using the difference of the two signals as the exogenous input to drive the system. By changing the feedback gain we could shape up the response to over damped, critically damped, under damped, or unstable behavior. Finally, we could use a pre-gain to adjust the DC gain of the closed loop to any desired value. This method of feedback gain tuning works well to locate the closed loop poles at any point on the root locus. However, the limt of the root locus design is that it does not have any ability to change the root locus itself to a desired shape.